Math 2010

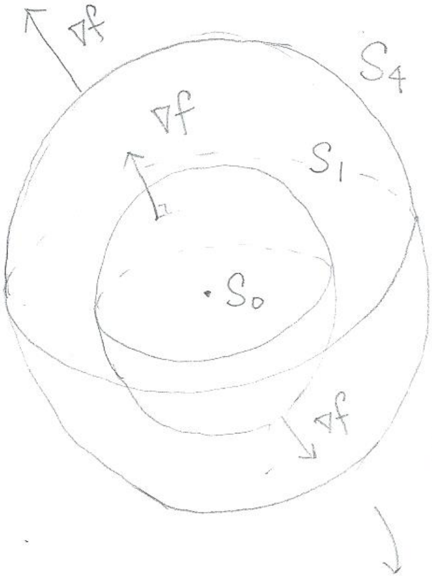
## § Lagrange Multipliers

Example: Let $f(x,y,z) = x^2 + y^2 + z^2$

Recall that: 1) $\nabla f \perp S_k$ for generic $k \geq 0$.

$S_k$ is the level surface of $f$.

$S_0 = \{(0,0,0)\}$, $S_1 = \{(x,y,z) \mid x^2+y^2+z^2\}$ see Pic. 1



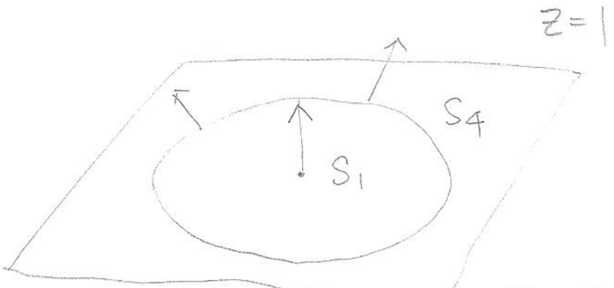

Pic. 1

Now, if we consider a constrant:

$f$ behave on $\{z=1\}$,

then these level surfaces will be a family of circles on $\{z=1\}$

See Pic. 2




Pic. 2

We can see, in this case $f$ attaches a minimum at $(0,0,1)$.

Meanwile, $\nabla f(0,0,1)$ is the only one vector which is perpendicular to $\{z=1\}$ plane., among all of $\nabla f(x,y,1)$.

In general, if we have a 3-variable function $f(x,y,z)$ with a constrant $g(x,y,z)=0$. Then $f$ attaches a maximum / minimum at $(x_0, y_0, z_0)$ on $\{g(x,y,z)=0\}$ only if $\nabla f(x_0, y_0, z_0)$ is a normal vector of $\{g(x,y,z)=0\}$ at $(x_0, y_0, z_0)$. So $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$ for some $\lambda \in \mathbb{R}$.

Define: We call $\lambda$ a Lagrange multiplier.

Rmk: $\lambda$ is also a variable we should solve when we want to find the extreme values. Since we have

$$\begin{cases} \nabla f(x_0, y_0, z_0) = \lambda g(x_0, y_0, z_0) \\ g(x_0, y_0, z_0) = 0 \end{cases}$$

4 variables & 4 eq, $(x_0, y_0, z_0)$, $\lambda$ are "solvable" "theoretically".

Thm: (Lagrange multipliers)

Suppose that $f(x,y,z)$ has a local max / min at $(x_0, y_0, z_0)$ on the surface $g(x,y,z)=0$. Then there exists $\lambda \in \mathbb{R}$ such that $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$

Proof: Let $(x(t), y(t), z(t)) = r(t)$ be a smooth curve on

$\{ g(x,y,z) = 0 \}$ and $r(0) = (x_0, y_0, z_0)$

So $f(r(t))$ attaches a local max / min at $t=0$.

$$\Rightarrow \frac{d}{dt} f(r(t)) \Big|_{t=0} = 0$$

$$= \frac{\partial f}{\partial x}(x_0, y_0, z_0) \cdot x'(0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)\, y'(0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)\, z'(0)$$

$$= \nabla f (x_0, y_0, z_0) \cdot r'(0)$$

So $\nabla f (x_0, y_0, z_0) \perp r'(0)$. —— $(*)$

We can choose any curve on $\{ g(x,y,z) = 0 \}$ satisfies $r(0) = (x_0, y_0, z_0)$. It will always satisfy $(*)$.

So $\nabla f (x_0, y_0, z_0)$ will be perpendicular to the tangent plane of $\{ g(x,y,z) = 0 \}$ at $(x_0, y_0, z_0)$.

So $\nabla f (x_0, y_0, z_0) = \lambda \nabla g (x_0, y_0, z_0)$ for some $\lambda \in \mathbb{R}$

$\square$

We also have 2-variables version :

<u>Thm</u>: If $f(x, y)$ has a local max/min at $(x_0, y_0)$. on the surface $g(x, y) = 0$. Then there exists $\lambda \in \mathbb{R}$ such that $\nabla f = \lambda \nabla g$ at $(x_0, y_0)$.

<u>Example</u> : $f(x, y) = xy$, $g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1$

$$\nabla f = (y, x). \quad \nabla g = \left(\frac{x}{4}, y\right)$$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \Rightarrow \begin{cases} y = \frac{\lambda x}{4}, \ x = \lambda y \\ \frac{x^2}{8} + \frac{y^2}{2} = 1 \end{cases}$$

$$\Rightarrow y = \frac{\lambda^2}{4} y \Rightarrow \lambda = \pm 2$$

when $\lambda = 2$,

$$\frac{x^2}{8} + \frac{y^2}{2} = \frac{4y^2}{8} + \frac{y^2}{2} = y^2 = 1$$

$$\Rightarrow y = \pm 1$$

$\therefore (x, y) = (2, 1), (-2, -1)$

when $\lambda = 2$, we also have $y = \pm 1$

$(x, y) = (-2, 1), (2, -1)$

$f(2,1) = 2 \qquad f(-2,1) = -2$

$f(-2,-1) = 2 \qquad f(2,-1) = -2$

$\downarrow \qquad\qquad \downarrow$

local max $\qquad$ local min.

## Lagrange Multipliers with two constraints:

If we have $f(x,y,z)$ with two constraints

$$\begin{cases} g_1(x,y,z) = 0 \\ g_2(x,y,z) = 0 \end{cases} \quad \text{where } \nabla g_1 \not\parallel \nabla g_2$$

Then $f$ has extreme value at $(x_0, y_0, z_0)$ on $\{g_1 = 0\} \cap \{g_2 = 0\}$

only if $\nabla f(x_0, y_0, z_0) = \lambda \nabla g_1(x_0, y_0, z_0) + \mu \nabla g_2(x_0, y_0, z_0)$

for some $\lambda, \mu \in \mathbb{R}$.

Proof: Let $\{r(t)\} = \{g_1 = 0\} \cap \{g_2 = 0\}$

$$r(0) = (x_0, y_0, z_0).$$

So we have

$$0 = \frac{d}{dt} f(r(t))\Big|_{t=0} = \nabla f(x_0, y_0, z_0) \cdot r'(0)$$

So $r'(0) \perp \nabla f(x_0, y_0, z_0)$

and $\begin{cases} r'(0) \perp \nabla g_1(x_0, y_0, z_0) \\ r'(0) \perp \nabla g_2(x_0, y_0, z_0) \end{cases}$ with $\nabla g_1 \not\parallel \nabla g_2 \Rightarrow \nabla f = \lambda \nabla g_1 + \mu \nabla g_2$ at $(x_0, y_0, z_0)$.

<u>Example:</u> Let $f(x,y,z) = 4x^2+4y^2+z^2$

$$\begin{cases} g_1 = x^2+y^2-1 \\ g_2 = x+y+z-1 \end{cases}$$

So $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$

$$\Rightarrow \begin{cases} (8x, 8y, 2z) = \lambda(2x, 2y, 0) + \mu(1,1,1) \\ x^2+y^2=1 \\ x+y+z=1 \end{cases}$$

$$\Rightarrow \begin{cases} (8-2\lambda)x = \mu \\ (8-2\lambda)y = \mu \\ 2z = \mu \\ x^2+y^2=1 \\ x+y+z=1 \end{cases}$$

$$x = \frac{\mu}{8-2\lambda}, \quad y = \frac{\mu}{8-2\lambda}, \quad z = \frac{\mu}{2}$$

$\Rightarrow x = y$, and $x^2+y^2=1 \Rightarrow x = \frac{\pm 1}{\sqrt{2}}$

$$\therefore (x,y,z) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\sqrt{2}\right)$$

$$\text{or } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1+\sqrt{2}\right)$$